1 Two peaks

1.1 Situation

A hiker determines the height of two adjacent mountain summits with his handheld GPS receiver, yielding h_1^{GPS} and h_2^{GPS}. During another tour, he wants to control the GPS heights with his barometric altimeter. Since he forgot to calibrate the sensor at a known height, he can only measure the height difference between the two summits, i.e., $\Delta h_{12}^{\text{Baro}}$.

1.2 Task

Determine the **two heights** (i.e., these are the unknowns), their **standard deviations**, and their **correlation coefficient** by the recursive least-squares approach (the initial estimate and covariance matrix follow from the GPS measurements). **Interpret** your findings.

1.3 Numerical data

Quantity	Value	Remarks
h_1^{GPS}	965 m	GPS height of summit 1
h_2^{GPS}	1055 m	GPS height of summit 2
σ_h^{GPS}	10 m	Assumed standard deviations of the GPS heights (the heights may be regarded as uncorrelated)
$\Delta h_{12}^{\text{Baro}}$	100.0 m	Barometric height difference
$\sigma_{\Delta h}^{\text{Baro}}$	1.0 m	Assumed standard deviation of the barometric height difference

1.4 Required equations

Recursive least-squares estimation

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{P}_0\,\mathbf{H}_1^T(\mathbf{H}_1\,\mathbf{P}_0\,\mathbf{H}_1^T + \mathbf{R}_1)^{-1}\,, \\ \hat{\mathbf{x}}_1 &= \hat{\mathbf{x}}_0 + \mathbf{K}_1(\mathbf{z}_1 - \mathbf{H}_1\,\hat{\mathbf{x}}_0)\,, \\ \mathbf{P}_1 &= (\mathbf{I} - \mathbf{K}_1\,\mathbf{H}_1)\mathbf{P}_0\,. \end{aligned}$$

Hint

Correlation coefficient

$$r_{12} = \frac{\sigma_{12}}{\sigma_1\,\sigma_2}\,.$$

Solution

Resolved in class on the black board.

2 Dam inspection with drones

You are asked to inspect the surface of a dam and check for potential cracks in the structure. The tilted section of the dam is flat and can be approximated as a 2D plane.

Close to the dam, GPS interference are very strong and you cannot rely on that signal to estimate the position of your drone accurately enough. You thus decide to place beacons broadcasting RFID[1] signals at know locations of the dam and to equip your drone with the corresponding receiver. This receiver allows you to measure the distance of your drone with respect to each of the broadcasting beacon. Halfway through the inspection, you detect a crack and want to accurately estimate its position.

2.1 Non linear least squares

The following information is available to you:

Approximate position of the drone, estimated at sight w.r.t. the first beacon: (x,y) = (65,15)

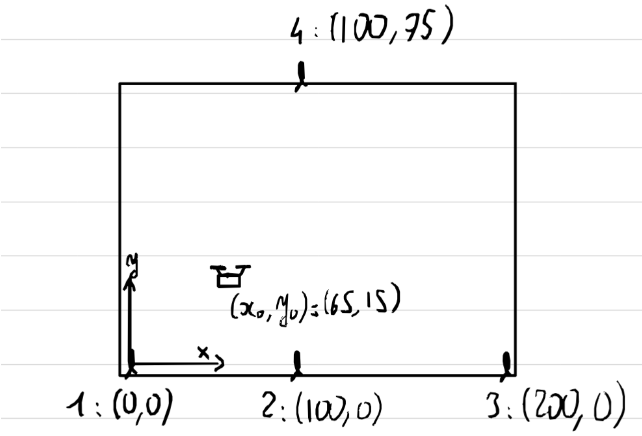
Beacon n°	Coordinates (x,y)	Distance to beacon (m)	Std of distance (m)
1	(0,0)	73.46	0.05
2	(100,0)	28.08	0.10
3	(200,0)	127.30	0.07

1. Draw a pictorial representation of the given problem.
2. What are the unknown variables that need to be estimated?
3. Write an equation relating the observations and variables to be estimated
4. Construct the **H** matrix[2]
5. Estimate the position of the drone using the least squares approach. Two iterations are sufficient to get a good enough estimate in this case.
6. What is the covariance matrix (**P**) of the estimate.
7. Why are we more confident in the position estimation over x axis compared the y axis ?

[1] Radio Frequency IDentification

[2] Hint: As the name suggests it is a non-linear LS; can you formulate Taylor's series expansion of the observation model up to the first order?

Solution



1.
2. The unknown position, $\mathbf{x}_r = \begin{bmatrix} x_r & y_r \end{bmatrix}^T$, of the drone needs to be estimated.
3.
$$\hat{z}_i = \sqrt{(x_r - x_i)^2 + (y_r - y_i)^2} \; \forall i \in \{1, 2, 3\}$$
4.
$$\mathbf{H}_i(x_0, y_0) = \frac{1}{\sqrt{(x_o - x_i)^2 + (y_o - y_i)^2}} \begin{bmatrix} x_o - x_i & y_o - y_i \end{bmatrix}$$
5. Since this is a non-linear least square, you will have to solve it in an iterative manner, starting with an initial guess provided (65 15). When utilizing the equations provided in the polycopie, it is essential to use the nonlinear function directly rather than its linear approximation when computing the error between the predicted and measured values. Employing the linear approximation to calculate the error could lead to the loss of crucial information. To further put it into words,

   Linear case: $\hat{\mathbf{x}}_j = \hat{\mathbf{x}}_{j-1} + \mathbf{K}_j \left(\mathbf{z}_j - \mathbf{H}_j \hat{\mathbf{x}}_{j-1} \right)$
   Non-linear case: $\hat{\mathbf{x}}_j = \hat{\mathbf{x}}_{j-1} + \mathbf{K}_j \left(\mathbf{z}_j - f(\hat{\mathbf{x}}_{j-1}) \right)$

   We estimate at each iteration the difference between the measured value (z_i) and estimated measure ($\hat{z}_i = \sqrt{(x_r - x_i)^2 + (y_r - y_i)^2}$) based on the latest estimate for the unknown. The **H** matrix is also recalculated. You get the solution after two iterations to be
   $\mathbf{x}_r = \begin{bmatrix} 73.011 & 8.059 \end{bmatrix}^T$
6. $\mathbf{P} = \begin{bmatrix} 0.0015 & -0.0003 \\ -0.0003 & 0.0546 \end{bmatrix}$
7. We can see that the uncertainty is higher along the y axis. This is because all three beacons are aligned at the bottom of the dam, hence causing positioning error along y axis to be less observable, lowering the confidence in the estimation

2.2 Recursive least squares

Additional beacon on top of the dam beacon that was previously offline suddenly transitions into an online state and starts to broadcast its signal. The following additional distance observation is available to the drone

Beacon n°	Coordinates (x,y)	Distance to Beacon (m)	Std of measure (m)
4	(100,75)	72.286	0.10

1. Construct the matrices needed to resolve this problem in a recursive manner ($\mathbf{H_0}$, $\mathbf{H_1}$, $\mathbf{P_0}$, $\mathbf{R_1}$)
2. What is the estimate of the position of the drone including this measurement ?
3. What is the new covariance matrix ($\mathbf{P_1}$) and how the standard deviation evolved considering the new measurement? Did the estimated covariance improve?

Solution

1. For measurement i,

$$f^i(\mathbf{x}_{cur}) = \sqrt{(x_{cur} - x^i)^2 + (y_{cur} - y^i)^2}$$
$$\mathbf{H}^i = \begin{bmatrix} \frac{x_{cur}-x^i}{\sqrt{(x_{cur}-x^i)^2+(y_{cur}-y^i)^2}} & \frac{y_{cur}-y^i}{\sqrt{(x_{cur}-x^i)^2+(y_{cur}-y^i)^2}} \end{bmatrix}$$

Since we have a solution from the previous example, we define $\mathbf{H}_0$ with the first three measures and $\mathbf{H}_1$ with the newest or the fourth measure. We define $\mathbf{P}_0$ using the solution from previous section. Hence,

$$\mathbf{H}_0 = \begin{bmatrix} 0.994 & 0.110 \\ -0.958 & 0.286 \\ -0.998 & 0.063 \end{bmatrix}$$
$$\mathbf{H}_1 = \begin{bmatrix} -0.374 & -0.987 \end{bmatrix}$$
$$\mathbf{P_0} = \begin{bmatrix} 0.0015 & -0.0003 \\ -0.0003 & 0.0546 \end{bmatrix}$$
$$\mathbf{R}_1 = \begin{bmatrix} 0.010 \end{bmatrix}$$

2. Solving the equations, we get

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 73.010 & 7.963 \end{bmatrix}$$

3. $\mathbf{P}_1 = \begin{bmatrix} 0.0015 & -0.0005 \\ -0.0005 & 0.0098 \end{bmatrix}$

From the values of the new covariance matrix, we can see that the values have reduced, showing increased confidence (certainty) in the final estimate. It is especially true for the uncertainty along y axis because the new beacon (on top of the dam) increases the observability of errors on the y axis compared to the first three beacons aligned at the bottom

3 (At home) Snow depth measurement, RLS

1. A drone is used for observing snow accumulation on a remote peak by determining its height in winter:

$$h_w = 2000.4 \text{ m}$$

from which it subtracts the height determined in summer:

$$h_s = 1999.6 \text{ m}$$

2. A mountain guide is sent to verify the snow height determination by performing a manual observation with an avalanche probe:

$$\Delta h_p = 100 \text{ cm}$$

just after the drone observation.

Task: Determine the snow height on the peak and its standard deviation via RLS, considering the uncertainties of observations.

Master Sensor Orientation

$$\sigma_{h_s} = \sigma_{h_w} = 0.1 \text{ m}$$

$$\sigma_{\Delta h_p} = 7 \text{ cm}$$

Solution

1. **Number of parameters:** One (snow depth)
2. **Initialization** (x_0, P_0)
   - Initial values follow from:
   $$x_0 = \text{difference of the drone observations} \quad \text{(scalar)}$$
   - P_0 is derived via covariance propagation (scalar):
   $$P_0 = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 10^2 & 0 \\ 0 & 10^2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 200 \text{ cm}^2$$
3. **Update from observation** z_1
   - z_1 = probe observation and its respective R (scalar)

Numerical Solution

$$\Delta h(x_1) = 96 \text{ cm}$$

$$\text{std}_{x_1} = 6.3 \text{ cm}$$